

Some Cohomology Classes in Principal Fiber Bundles and Their Application to Riemannian Geometry

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ABSTRACT We define some new global invariants of a fiber bundle with a connection. They are cohomology classes in the principal fiber bundle that are defined when certain characteristic curvature forms vanish. In the case of the principal tangent bundle of a riemannian manifold, they are invariant under a conformal transformation of the metric. They give necessary conditions for conformal immersion of a riemannian manifold in euclidean space.

1. INTRODUCTION

In this announcement, we discuss some real cohomology classes that arise in the total space of a principal fiber bundle. These classes, in the cases when they are interesting, depend on a connection in that bundle. We consider, in particular, the principal bundle of a riemannian manifold with the natural riemannian connection. We show that these classes depend only on the conformal structure of the manifold, and moreover that their integrality is a necessary condition for global conformal immersion in euclidean space with certain prescribed codimension.

2. A SPECIAL CASE—RIEMANNIAN 3-MANIFOLDS

To illustrate these ideas, we consider a special case. Let M be a compact, oriented, 3-dimensional riemannian manifold. Let $F(M)$ be the $SO(3)$ bundle over M consisting of oriented, orthonormal tangent frames. Let $\theta = (\theta_{ij})$ and $\Omega = (\Omega_{ij})$ denote the connection and curvature forms associated to the riemannian connection in $F(M)$. θ is a 3×3 skew symmetric matrix of 1-forms on $F(M)$, and Ω is a 3×3 skew symmetric matrix of 2-forms on $F(M)$. They are related by the structural equation

$$(2.1) \quad d\theta_{ij} = \Omega_{ij} - \sum_{k=1}^3 \theta_{ik} \wedge \theta_{jk}.$$

Using these forms, we define Q , a differential 3-form on $F(M)$ by

$$Q = \frac{1}{8\pi^2} \left[\theta_{12} \wedge \theta_{13} \wedge \theta_{23} - \theta_{12} \wedge \Omega_{12} - \theta_{13} \wedge \Omega_{13} - \theta_{23} \wedge \Omega_{23} \right].$$

Lemma 2.2. Q is a closed form.

Proof. Direct from 2.1 by calculation.

Lemma 2.3. $\int_{F(M)_m} Q = 1$, where $F(M)_m$ is the fiber over any $m \in M$.

Proof. $F(M)_m$ is equivalent to $SO(3)$, and since Ω is horizontal, $Q|_{F(M)_m}$ is a non-zero constant multiple of the volume form on $SO(3)$. The constant $1/8\pi^2$ was chosen to normalize this integral.

Lemma 2.4. Let χ, χ' be two global cross sections in $F(M)$. Then $\int_{\chi} Q - \int_{\chi'} Q \in \mathbb{Z}$.

Proof. Since M is an oriented 3-manifold, $F(M)$ is trivial. Thus, up to torsion $\chi' = \chi + nF(M)_m$ where $n \in \mathbb{Z}$. The result then follows from Lemma 2.3.

Definition. Let $\lambda: R \rightarrow R/Z$ be the natural homomorphism. We define $\Phi(M) \in R/Z$ by $\Phi(M) = \lambda(\int_{\chi} Q)$, where χ is any global cross section. Lemma 2.4 shows that Φ is well defined.

Thus, associated to any riemannian 3-manifold, M , is a point $\Phi(M) \in S^1$. Two important properties of Φ are the following:

Theorem 2.5. $\Phi(M)$ is a conformal invariant. That is, if \bar{M} is conformally equivalent to M , then $\Phi(\bar{M}) = \Phi(M)$.

Theorem 2.6. A necessary condition that M admit a global conformal immersion in R^4 is that $\Phi(M) = 0$.

Generalizations of these theorems appear in §5.

Example 2.7. Regard S^3 as a Lie group. Given any $y \in R/Z$ we can find a left invariant metric, g , on S^3 such that $\Phi(S^3, g) = y$.

Example 2.8. Let RP^3 denote real projective 3-space together with its canonical metric. Then $\Phi(RP^3) = 1/2$. Thus there exists no global conformal immersion of RP^3 into R^4 .

The second example shows that $\Phi(M)$ is a global geometric obstruction to immersability, since locally RP^3 is isometrically imbeddable in R^4 , and globally RP^3 is C^∞ immersible in R^4 .

A property of Φ that seems special to 3-manifolds is its behavior as a map from conformal structures into the circle. In fact, let m denote the space of conformal

structures on a fixed, compact, oriented manifold M . Then $\Phi: m \rightarrow S^1$.

Theorem 2.9. $g \in m$ is a critical point of Φ if and only if (M, g) is locally conformally flat.

Corollary 2.10. Suppose M is simply connected. Then either Φ has exactly one critical point and M is diffeomorphic to S^3 , or Φ has no critical points and M is not diffeomorphic to S^3 .

3. PRINCIPAL G-BUNDLES

Let G be a Lie group with Lie algebra \mathfrak{g} . Let M be a compact manifold and $E \xrightarrow{\pi} M$ a principal fiber bundle over M with fiber G . Let θ be a connection on E with curvature Ω . We recall that θ is a vertical \mathfrak{g} -valued 1-form on E , Ω is a horizontal \mathfrak{g} -valued 2-form on E , and both are equivariant under right action by G . They are related by the structural equation

$$(3.1) \quad d\theta = \Omega - \frac{1}{2}[\theta, \theta],$$

where $[\theta, \theta](x, y) = [\theta(x), \theta(y)]$.

Let $I^l(\mathfrak{g})$ denote the space of polynomials of degree l on \mathfrak{g} which are invariant under the adjoint action of G . Set $I(\mathfrak{g}) = \Sigma \oplus I^l(\mathfrak{g})$. $I(\mathfrak{g})$ is a graded algebra. If ψ is an equivariant differential form on E taking values in $\mathfrak{g}^i = \mathfrak{g} \otimes \dots \otimes \mathfrak{g}$ and if $P \in I^l(\mathfrak{g})$, then $P(\psi) = P \circ \psi$ is a real-valued form on E invariant under right action by G . If ψ is a p -form on E taking values in \mathfrak{g}^i and φ is a q -form on E taking values in \mathfrak{g}^s then in a natural way $\psi \wedge \varphi$ is a $(p+q)$ -form on E taking values in \mathfrak{g}^{i+s} .

In particular $\Omega^i = \Omega \wedge \dots \wedge \Omega$ is a $2l$ -form on E taking values in \mathfrak{g}^i . Thus if $P \in I^l(\mathfrak{g})$ then $P(\Omega^i)$ is a real-valued, invariant, horizontal, $2l$ -form on E . For background on these notions the reader should consult [1]: We recall the well-known

Theorem 3.2 (Weil-homomorphism). Let $P \in I^l(\mathfrak{g})$. Then $P(\Omega^i)$ is closed, and there is a unique closed $2l$ -form $\overline{P}(\Omega^i)$ on M such that $\overline{P}(\Omega^i) \circ d\pi = P(\Omega^i)$. Moreover, the element of $H^{2l}(M, \mathbb{R})$ defined by $\overline{P}(\Omega^i)$ is independent of the connection, θ . Finally, the map thus defined from $I(\mathfrak{g}) \rightarrow H^*(M, \mathbb{R})$ is an algebra homomorphism.

We begin with the observation that not only is $P(\Omega^i)$ closed in E but it is exact. In fact, for $t \in [0, 1]$ set $\Omega_t = t\Omega + \frac{1}{2}(t^2 - t)[\theta, \theta]$. Then define

$$TP(\theta) = l \int_0^1 P(\theta \wedge \Omega_t^{i-1}) dt.$$

$TP(\theta)$ is a real-valued, invariant, $(2l-1)$ -form on E , and

Lemma 3.3. $dTP(\theta) = P(\Omega^i)$.

Proof. Straightforward calculation.

Lemma 3.4. Let $E \rightarrow M$ and $\hat{E} \rightarrow \hat{M}$ both be principal G bundles and let $\varphi: E \rightarrow \hat{E}$ be a bundle map. Let θ be

a connection on E , $\hat{\theta}$ a connection on \hat{E} , and suppose that $\hat{\theta} \circ d\varphi = \theta$. Then $P(\Omega^i) = P(\hat{\Omega}^i) \circ d\varphi$ and $TP(\theta) = TP(\hat{\theta}) \circ d\varphi$.

Proof. Since $\hat{\Omega} \circ d\varphi = \Omega$ the statements follow immediately.

It is reasonable to ask whether T is the only such operator. In fact, it is not, but if one insists on naturality it is unique up to an exact remainder.

Lemma 3.5. Let T' be a map that assigns to each $P \in I^l(\mathfrak{g})$ a differential $2l-1$ form $T'P(\theta)$ on each principal G bundle with connection θ . Suppose T' satisfies:

$$(3.6) \quad dT'P(\theta) = P(\Omega^i).$$

$$(3.7) \quad T' \text{ is natural in the sense of Lemma 3.4.}$$

Then for any bundle E , $T'P(\theta) = TP(\theta) + \text{exact}$.

Proof. Let $E_G \rightarrow M_G$ be the universal bundle and classifying space of G . Now E_G is asyclic and so, since $dT'P(\theta) = P(\Omega^i) = dTP(\theta)$, we must have $T'P(\theta) = TP(\theta) + \text{exact}$ for any connection on E_G . Since any connection on E may be obtained as the pull back of one on E_G under a bundle map, the lemma follows from Lemma 3.4.

Since $I(\mathfrak{g})$ is an algebra, it is useful to have

Lemma 3.8. If $P \in I^l(\mathfrak{g})$ and $Q \in I^s(\mathfrak{g})$ then

- 1) $PQ(\Omega^{i+s}) = P(\Omega^i) \wedge Q(\Omega^s)$;
- 2) $TPQ(\theta) = TP(\theta) \wedge Q(\Omega^s) + \text{exact}$.

Proof. 1) is a direct calculation. 2) may be easily proved on the universal bundle and then pulled back to E . In fact, for any connection θ on E_G

$$d(TP(\theta) \wedge Q(\Omega^s)) = P(\Omega^i) \wedge Q(\Omega^s) = PQ(\Omega^{i+s}) = d(TPQ(\theta)).$$

Thus 2) is true on E_G by asyclicity.

Definition. Let $P \in I^l(\mathfrak{g})$ such that $P(\Omega^i) \equiv 0$. Then $dTP(\theta) = 0$, and so $TP(\theta)$ defines an element $TP[\theta] \in H^{2l-1}(E, \mathbb{R})$. Note that by Lemma 3.5 this class is independent of which natural operator, T , we are using.

Example 1. Suppose $2l > \dim M$. Then $P(\Omega^i) \equiv 0$ since it is a horizontal $2l$ -form and the dimension of the horizontal space equals $\dim M$. Thus $TP[\theta] \in H^{2l-1}(E, \mathbb{R})$ is defined. However, the class may not be independent of connection.

Lemma 3.9. Let $\theta(s)$ be a smooth 1-parameter family of connections on E . Set $\theta' = d/(ds)(\theta(s))|_{s=0}$. Then θ' is a horizontal \mathfrak{g} -valued 1-form on E and

$$\frac{d}{ds}(TP(\theta(s)))|_{s=0} = lP(\theta' \wedge \Omega^{i-1}) + \text{exact}.$$

Proof. This may be calculated directly, or one may prove the formula first on the universal bundle and pull it back to E . In the latter case, the calculation is simplified by asyclicity.

Lemma 3.10. Suppose $2l = \dim M + 1$; then $TP[\theta]$ depends on θ . Suppose $2l > \dim M + 1$; then $TP[\theta]$ is independent of θ .

Proof. This follows directly from Lemma 3.9 since θ' is horizontal and thus $P(\theta' \wedge \Omega^{l-1})$ is a horizontal $2l - 1$ form, which must be 0 if $2l > \dim M + 1$ and in general is not exact for $2l = \dim M + 1$.

So far we have only seen the classes $TP[\theta]$ defined when $P(\Omega^l)$ vanished for reasons of dimension. In the next section it is shown that there are sometimes geometric reasons why lower dimensional $P(\Omega^l) \equiv 0$, and in these cases, as well as when $2l = \dim M + 1$, the classes $TP[\theta]$ provide global information about the connection.

4. $GL(n, R)$ BUNDLES

Let $gl(n, R)$ be the Lie algebra of the full linear group, $Gl(n, R)$. It consists of all $n \times n$ matrices. Let $[n/2]$ denote the smallest integer $\leq n/2$. For $i = 1, \dots, [n/2]$, let $P_i \in I^{2i}$ be the i th pontrjagin polynomial, normalized so that the corresponding cohomology class is integral, i.e., for any $Gl(n, R)$ bundle E over M with connection having curvature Ω , the form $P_i(\Omega^{2i})$ lives over the i th real pontrjagin class of F , normalized to be in $H^{4i}(M, Z)$. We also define the inverse pontrjagin polynomials, P_i^\perp ,

$$P_i^\perp = -P_i - P_{i-1}P_1^\perp - \dots - P_1P_{i-1}^\perp,$$

$$i = 1, 2, \dots$$

These polynomials also lead to integral classes in M and are, in fact, the pontrjagin classes of the inverse of the vector bundle associated to E .

Lemma 4.1. Let E be a $Gl(n, R)$ bundle with connection θ . Then $TP_i(\theta)|Gl(n, R)$ and $TP_i^\perp(\theta)|Gl(n, R)$ are closed. If we set y_i = cohomology class determined by $TP_i(\theta)|Gl(n, R)$ then $TP_i^\perp(\theta)|Gl(n, R) \in -y_i$.

Proof. Obviously their restriction to the fiber is closed since their differential is horizontal. The second statement is immediate from Lemma 3.8.

$y_i \in H^{4i-1}(Gl(n, R), R)$. For n odd, it is well known that $y_1, \dots, y_{[n/2]}$ are non-zero and $H^*(Gl(n, R), R)$ is isomorphic to the grassmann algebra with these as generators. For n even, $y_{[n/2]} = 0$, and $H^*(Gl(n, R), R)$ is generated by the remaining y_i and an $(n - 1)$ class, x .

Lemma 4.2. $1/2y_i \in H^{4i-1}(Gl(n, R), Z)$.

Proof. This is well known. In fact y_i is the transgression of p_i , the i th pontrjagin class of the classifying space. Since p_i is integral, y_i is integral. Moreover y_i reduced mod 2 is the transgression of $W_{2i} \cup W_{2i}$, the

square of the $2i$ -th Stiefel-Whitney class. But the transgression of a product is 0. For background see [2].

We thus have

Lemma 4.3. If E is a principal $Gl(n, R)$ bundle with connection θ , and $P_i(\Omega^{2i}) \equiv 0$, then $1/2TP_i[\theta]|Gl(n, R) \in H^{4i-1}(Gl(n, R), Z)$. The same is true for P_i^\perp .

In general, when $TP_i[\theta]$ or $TP_i^\perp[\theta]$ exist, there is no reason to expect it, or half of it, to be an integral class in all of E . However, this is sometimes the case.

Example 2. Let $G_{n,k}$ denote the grassmann manifold of oriented n -planes in R^{n+k} . Let $E_{n,k} \rightarrow G_{n,k}$ be the canonical $Gl(n, R)$ bundle. Points in $E_{n,k}$ are $(n + 1)$ -tuples $(H; b_1, \dots, b_n)$, where $H \in G_{n,k}$ and b_1, \dots, b_n is a basis of H . $E_{n,k}$ is equipped with a natural connection θ , with respect to which parallel translation preserves the natural inner product in the associated n -dim vector bundle. The importance of this connection lies in the following:

Fact 4.4. Let M be an n -dimensional riemannian manifold and let $B(M)$ be the basis bundle of M . $B(M)$ is a $Gl(n, R)$ bundle equipped with the unique riemannian connection $\bar{\theta}$. Let $h: M \rightarrow R^{n+k}$ be an isometric immersion, and let $f_h: M \rightarrow G_{n,k}$ be the gauss map. Then f_h is covered by a natural bundle map $F_h: B(M) \rightarrow E_{n,k}$ and $\theta \circ dF_h = \bar{\theta}$.

Theorem 4.5. Let θ be the standard connection on $E_{n,k}$. Then for $[k/2] + 1 \leq i \leq [(n - 1)/2]$

- 1) $P_i^\perp(\Omega^{2i}) \equiv 0$;
- 2) $1/2TP_i^\perp[\theta] \in H^{4i-1}(E_{n,k}, Z)$.

Proof. Since the vector bundle associated to $E_{n,k}$ has a k -dimensional inverse, we know that the cohomology class in $G_{n,k}$ corresponding to $P_i^\perp(\Omega^{2i})$ is 0 for i in this range. Because θ is the canonical connection and because $G_{n,k}$ is a symmetric space, it is easy to show that the form $P_i^\perp(\Omega^{2i})$ in $G_{n,k}$ that represents this class is invariant on $G_{n,k}$. But on a symmetric space an invariant, exact form is identically 0. Thus $P_i^\perp(\Omega^{2i}) = P_i^\perp(\Omega^{2i}) d\pi \equiv 0$. This shows 1). The proof that $1/2TP_i^\perp[\theta]$ is an integral class is fairly lengthy and requires an analysis of the cohomology ring of $E_{n,k}$ and the consideration of complex grassmann and Stiefel manifolds.

5. CONFORMAL INVARIANCE AND CONFORMAL IMMERSIONS

Let M be an n -dimensional manifold with basis bundle $B(M)$.

Theorem 5.1. Let g_1, g_2 be conformally equivalent riemannian metrics on M and let θ_1, θ_2 be the corresponding riemannian connections on $B(M)$. Let Ω_1, Ω_2 denote the respective curvature forms. Then for any l and any $P \in I^l(gl(n, R))$

$$TP(\theta_1) = TP(\theta_2) + \text{exact}.$$

Corollary 5.2. Let M be a riemannian manifold. Let θ, Ω be the riemannian connection and curvature forms on $B(M)$. Then for any P

- 1) $P(\Omega^i)$ is a conformal invariant;
- 2) $TP[\theta]$ is a conformal invariant when $P(\Omega^i) \equiv 0$.

Proof. The corollary is immediate from the theorem. The proof of the theorem follows from Lemma 3.9 and a fairly long calculation. Use is made of the fact that the invariant polynomials on $gl(n, R)$ are generated by $\text{tr} A^i$.

Theorem 5.3. Let M be a riemannian manifold. Let θ, Ω be the riemannian connection and curvature forms on $B(M)$. Then a necessary condition that M be locally conformally immersible in R^{n+k} is that $P^\perp_i(\Omega^{2i}) \equiv 0$ for $[k/2] + 1 \leq i \leq [(n-1)/2]$.

Theorem 5.4. A necessary condition that M admit a global conformal immersion in R^{n+k} is that $P^\perp_i(\Omega^{2i}) \equiv 0$ and that $\frac{1}{2}TP_i[\theta]$ be an integral class for $[k/2] + 1 \leq i \leq [(n-1)/2]$.

The proof of these theorems is immediate from Corollary 5.2, Theorem 4.5, and Fact 4.4.

Remark 5.5. Since $P^\perp_i(\Omega^{2i})$ is a $4i$ form, we know for dimension reasons that $P^\perp_i(\Omega^{2i}) \equiv 0$ for $i > n/4$. Moreover, from Lemma 3.10 we know that for $i > (n+1)/4$, $TP^\perp_i[\theta]$ is independent of connection. In fact, it may be shown that for $i > (n+1)/4$, $\frac{1}{2}TP^\perp_i[\theta]$ is always an integral class. Thus Theorem 5.3 and 5.4 are important only for i in the range $[k/2] + 1 \leq i \leq (n+1)/4$. In other words, these theorems are of interest for conformal immersions in codimensions $\leq n/2$.

Remark 5.6. The theorems for riemannian manifolds have been done in the context of the $Gl(n, R)$ basis bundle rather than in the $O(n)$ frame bundle. This was primarily because the frame bundle changes as the

metric changes and the basis bundle does not. It is thus awkward to talk about conformally invariant forms and classes in a changing bundle. In fact, it makes no difference in that the basis bundle is a deformation retract of the frame bundle. Moreover, the polynomials P_i and P^\perp_i exist as well on the Lie algebra of skew symmetric matrices and the forms and classes $P_i(\Omega^{2i})$ and $TP_i[\theta]$, etc. are defined on the frame bundle and have the same existence and integrality properties.

Remark 5.7. Let M be a $(4k-1)$ -dimensional riemannian manifold, and let $P = P(P_1, P_2, \dots, P_k)$ be a polynomial of degree $2k$. Then $P(\Omega^{2k}) \equiv 0$ and $TP[\theta] \in H^{4k-1}(B(M), R)$. These classes are conformal invariants of M . It is the only case where the class always exists and yet depends on the metric. For, as we have seen, if $\deg P > 2k$, $TP[\theta]$ is independent of connection and (probably) uninteresting. For $\deg P < 2k$, in general $TP(\theta)$ does not exist. One actually needs $\dim M = 4k-1$, and not simply odd, since it can be easily shown that the only polynomials that lead to non-trivial classes are already polynomials in the $\{P_i\}$.

Thus the same polynomials that give pontrjagin numbers for compact $4k$ -dimensional C^∞ manifolds give conformally invariant classes in $B(M)$ for $(4k-1)$ -dimensional riemannian manifolds.

Remark 5.8. In Section 2... we gave the example of a compact riemannian 3-manifold. The class in $F(M)$ defined by the form, Q , is exactly the class $\frac{1}{2}TP_1[\theta]$, where P_1 is interpreted as a polynomial on the Lie algebra of $O(n)$ rather than $Gl(n, R)$.

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